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Banach Spaces of Weights on Quasimanuals

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Section 1 is a brief introduction. Section 2 contains the basic definitions of quasimanuals, weights, and operational logics. The linear space $\mathcal{W}$ of all weights on a quasimanual $\mathcal{A}$ is introduced and given a norm. $\mathcal{W}$ with this norm is seen to be a Banach space. The subspace $\mathcal{V}$ of $\mathcal{W}$ generated by the positive cone of $\mathcal{W}$ is given the base norm and is also shown to be an Archimedian ordered Banach space with an additive norm. In Section 3 normal linear functionals on $\mathcal{V}^*$ are defined in analogy with normal linear functionals on w^* algebras. The space $\mathcal{V}$ is shown to be the set of normal functionals on $\mathcal{V}^*$ and we show $\mathcal{V}$ to be the unique partially ordered Banach space with a closed generating cone which is predual to $\mathcal{V}^*$. Next, weakly compact subsets of $\mathcal{W}$ are characterized in terms of eventwise convergence. This is the Hahn-Vitali-Saks theorem of classical measure theory in this noncommutative setting; several weak compactness results are drawn from this and compared with their classical counterparts. Section 4 introduces the ultraweak topology for $\mathcal{V}^*$ in analogy with the same for the trace class operators on Hilbert space. Here the condition for a compact base for the cone of $\mathcal{V}$ is examined and shown to be a poor and unnecessary hypothesis in many circumstances. Many connections with the existent literature are made and throughout the paper there are many examples and open questions.

1. INTRODUCTION

At the 25^e Cours de perfectionnement de l'Association Vaudoise des Chercheurs en Physique held in Montana, Switzerland in March 1983, Randall and Foulis presented a paper entitled "A mathematical language for quantum physics" (Randall and Foulis, 1983). This language takes into account the operational, realistic, probabilistic, and subjective approaches to quantum mechanics and is based, quite simply, on set theory. It is our purpose to describe the various Banach spaces which naturally arise in this theory. We need only the most elementary parts of the language developed in Randall and Foulis (1983) and these are introduced below. Very few

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physical axioms are made about the quasimanuals defined and described there, as our purpose is to establish a number of elementary properties of these Banach spaces without extraneous physical assumptions. For example, no use is made of the more refined notion of *manual* introduced by Foulis and Randall; we do not interpret the elements of a quantum logic as filters acting on states of a *physical system*; even the well-known orthomodular identity plays no explicit role here. As the reader will see, the mathematical conclusions which can be deduced in this simple setting are rather extensive. Considerable use will be made of weak topologies and ordered normed spaces, so we have adopted the general notation of Kelley and Namioka (1963) and refer the reader to this well-known work frequently. Base-normed and order-unit-normed spaces are used extensively, so Alfsen's monograph (1971) is also referred to very often. Some proofs are perhaps a bit more complete than necessary but we felt this more instructive than disseminating a road map through several very different functional analysis texts.

2. BASIC DEFINITIONS AND BANACH SPACES OF WEIGHTS

Let X be a nonempty set and let $\mathscr{A}$ be a nonempty collection of nonempty subsets of X. The collection $\mathscr{A}$ is called a *quasimanual* on X when $X = \bigcup_{E \in \mathscr{A}} E$. The set X is called the set of *outcomes* for $\mathscr{A}$ and each E in $\mathscr{A}$ is called an *operation*. If $A \subset E \in \mathscr{A}$ then A is called an *event* and the set of all events for $\mathscr{A}$ is denoted $\mathscr{E}(A)$. We may think of the operations as experiments performed on a *physical system*, the outcomes are simply our observations, and the quasimanual is our given catalog of experiments. If A and B are disjoint events and $A \cup B \subset E \in \mathscr{A}$ for some operation E, then we say A and B are *orthogonal* and we write $A \perp B$. If $A \perp B$ and $A \cup B = E \in \mathscr{A}$, then we say A and B are *operational complements*; this we denote as A *oc* B. We now introduce a partial order and equivalence relation on $\mathscr{E}(\mathscr{A})$ whose interpretations are, respectively, implication and logical equivalence. Suppose A, B, and C are events with A *oc* C and B *oc* C (i.e., A and B share a common complement C); we call A and B *operationally perspective* and write A *op* B. If there exists a finite sequence of events $D_0, D_1, \ldots, D_n$ with $A = D_0$, $B = D_n$, and either $D_{i-1} \subset D_i$ or D_{i-1} *op* D_i for $i = 1, 2, \ldots, n$, then we say A *implies* B and we write $A \leq B$. Note that $A \leq A$, and $A \leq B$ and $B \leq C$ implies $A \leq C$. If we have events A and B such that $A \leq B$ and $B \leq A$ then we say that A and B are *logically equivalent* and we write $A \equiv B$. This equivalence relation partitions $\mathscr{E}(\mathscr{A})$ into disjoint classes called *operational propositions*: specifically, if $A \in \mathscr{E}(\mathscr{A})$ we write $p(A) = \{B \in \mathscr{E}(\mathscr{A}) \colon A \equiv B\}$ and call $p(A)$ the operational proposition corresponding to the event A. We designate with $\Pi(\mathscr{A})$, or simply Π, the set of all operational propositions associated with the quasimanual Π. The object

$\Pi(\mathscr{A})$ is what Foulis and Randall call an *operational logic* and it must be emphasized here that it was constructed without any preconceived stochastic notions.

We now introduce stochastic notions on our quasimanual. Let $\omega: X \to [0, 1] \subset \mathbb{R}$; then ω is called a *weight* on $\mathscr{A}$ provided for each E in $\mathscr{A}$, $\Sigma_{x \in E}\omega(x) = 1$. Since the range of ω is contained in $[0, 1]$, this last sum is understood to mean unordered summation. The set of all weights on $\mathscr{A}$ is denoted Ω and when Ω is nonempty, it is convex. For each event A, we define $\omega(A) = \Sigma_{x \in A}\omega(x)$. We shall abbreviate the notation $\Sigma_{x \in A}$ to Σ_A. Let $\mathbb{R}^X$ designate all the real valued functions on X.

1. Definition. Let $\mathscr{W} \subset \mathbb{R}^X$ be the set of functions satisfying the following two criteria: (i) For each $\mu \in \mathscr{W}$, $\sup\{\Sigma_E|\mu(x)|: E \in \mathscr{A}\} = M_\mu < \infty$; (ii) for each $\mu \in \mathscr{W}$, there exists a constant K_μ such that $\Sigma_E\mu(x) = K_\mu$ for each E in $\mathscr{A}$. Note: $\Sigma_E\mu(x)$ in (ii) is an unordered sum of real numbers whose existence is guaranteed by (i) and K_μ in (ii) depends only on μ and not on E in $\mathscr{A}$.

Since $\mathbb{R}^X$ is a partially ordered vector space and since $\mathscr{W}$ is clearly a linear subspace, we can restrict the partial order of $\mathbb{R}^X$ to $\mathscr{W}$. When the set Ω is nonempty it is contained in the positive cone $\mathscr{C} = \{\mu \in \mathscr{W}: \mu \geq 0\}$ since for each $\omega \in \Omega$, $M_\omega = K_\omega = 1$.

2. Proposition. If $\mu \in \mathscr{C}$ and $\mu \neq 0$, then there exist unique $\omega \in \Omega$ and scalar $\alpha > 0$ such that $\mu = \alpha\omega$ [i.e., Ω is a base for $\mathscr{C}$ (Alfsen, 1971)].

Proof. If $\mu > 0$, there exist $x_0 \in X$ and $E \in \mathscr{A}$ with $x_0 \in E$ and $K_\mu = \Sigma_E\mu(x) \geq \mu(x_0) > 0$. Clearly, $\omega = (1/K_\mu)\mu \in \Omega$, or $\mu = K_\mu\omega$. If $\mu = \alpha\omega = \beta\nu$ for positive scalars α, β and $\omega, \nu \in \Omega$, then for each $E \in \mathscr{A}$, $\alpha \cdot 1 = \alpha\omega(E) = \beta\omega(E) = \beta \cdot 1$. Thus, $\alpha = \beta$ and $\omega = \nu$.

Let $\mathscr{W}'$ represent the algebraic dual of $\mathscr{W}$. Each event $A \in \mathscr{E}(\mathscr{A})$ and, in particular, each outcome $x \in X$, generates a unique positive linear functional in $\mathscr{W}'$ as follows: For $A \in \mathscr{E}(\mathscr{A})$, let $f_A: \mathscr{W} \to \mathbb{R}$ be defined by $f_A(\mu) = \Sigma_A\mu(x) = \mu(A)$. Since $\mathscr{W} \subset \mathbb{R}^X$, there is no ambiguity in this definition and clearly each f_A is linear and positive. The functional f_A is called the *frequency functional* for the event A and we let $\mathscr{L}(\mathscr{A})$, or simply $\mathscr{L}$, designate $\{f_A: A \in \mathscr{E}(\mathscr{A})\}$ in $\mathscr{W}'$. To emphasize the dual roles of the signed weights in $\mathscr{W}$ and the functionals in $\mathscr{W}'$, we will write $\langle\mu, f\rangle$ for $f(\mu)$ when $\mu \in \mathscr{W}$ and $f \in \mathscr{W}'$, in particular, $\mu(A) = f_A(\mu) = \langle\mu, f_A\rangle$. If $x \in X$ then $\{x\} \in \mathscr{E}(\mathscr{A})$ and for simplicity we write f_x as the functional of $\{x\}$. If $E, F \in \mathscr{A}$ then $\langle\mu, f_E\rangle = \langle\mu, f_F\rangle = K_\mu$ for each $\mu \in \mathscr{W}$, i.e., $f_E = f_F$ in $\mathscr{W}'$. Let $e \in \mathscr{W}'$ represent f_E for each $E \in \mathscr{A}$. In particular, $\langle\omega, e\rangle = \omega(E) = 1$ for all $E \in \mathscr{A}$ and $\omega \in \Omega$. If $A \in \mathscr{E}(\mathscr{A})$ and $A \subset E \in \mathscr{A}$, then $f_{E\setminus A} = e - f_A$ in $\mathscr{W}'$. Observe that $f_{E\setminus A} = f_{F\setminus A}$ for any operations $E, F \in \mathscr{A}$ for which $A \subset E$ and $A \subset F$. Therefore, we call $e - f_A$ the *negation* of f_A and we write $e - f_A = f'_A$. Note that $f_A - f'_A = 2f_A - e$ in $\mathscr{W}'$.

A partially ordered linear space is said to be *Archimedean ordered* if for vectors x, y with $y \geq 0$ and $\alpha x \leq y$ for all $\alpha > 0$ implies $x \leq 0$. A cone $\mathscr{C}$ in a real, partially ordered, normed space is called a *normal* cone provided there exists a scalar $\alpha > 0$ such that for any $x, y \in \mathscr{C}$ with $\|x\| \geq 1$ and $\|y\| \geq 1$ implies $\|x+y\| \geq \alpha$. If $\mathscr{U}$ is the norm closed unit ball, $\mathscr{C}$ is normal iff $(\mathscr{U} + \mathscr{C}) \cap (\mathscr{U} - \mathscr{C})$ is bounded; further, the normality of $\mathscr{C}$ is equivalent to the existence of a topologically equivalent norm $\|\cdot\|_1$ such that $x \leq y$ in $\mathscr{C}$ implies $\|x\|_1 \leq \|y\|_1$. We call this norm $\|\cdot\|_1$ a *monotone* norm. For these equivalences one can consult, for instance (Kelly et al., 1963, pp. 226, 227). We say the norm $\|\cdot\|$ is *additive* when $\|x+y\| = \|x\| + \|y\|$ for $x, y \geq 0$. Note that an additive norm is monotone:

$$0 \leq x \leq y \Rightarrow y - x = z \geq 0 \Rightarrow \|y\| = \|x\| + \|z\| \geq \|x\|$$

3. Definition. For each $\mu \in \mathscr{W}$, let

$$\begin{aligned}\|\mu\| &= \sup\{\langle \mu, f_A - f'_A \rangle : A \in \mathscr{E}(\mathscr{A})\} \\ &= \sup\{2\langle \mu, f_A \rangle - \langle \mu, e \rangle : A \in \mathscr{E}(\mathscr{A})\}\end{aligned}$$

4. Theorem. The function $\|\cdot\|$ of Definition 3 is a norm on $\mathscr{W}$ and $\mathscr{W}$ with this norm is a Banach space. When $\Omega \neq \phi$, $\mathscr{W}$ is Archimedean ordered, each $\omega \in \Omega$ has unit norm, the cone $\mathscr{C}$ is closed in the weak topology $w(\mathscr{W}, \mathscr{L})$ and thus norm closed, and the norm is additive on $\mathscr{C}$ so $\mathscr{C}$ is a normal cone.

Proof. For each $\mu \in \mathscr{W}$, $\|\mu\| \geq \pm\langle \mu, e \rangle$; hence, $\|\mu\| \geq 0$. Next, $\|\mu\| < \infty$; for each $A \in \mathscr{E}(\mathscr{A})$, $A \subset E \in \mathscr{A} \Rightarrow |\Sigma_A \mu(x)| \leq \Sigma_E |\mu(x)| \leq M_\mu$. Therefore, $|\langle \mu, f_A - f'_A \rangle| \leq 2M_\mu$ or $\|\mu\| \leq 2M_\mu$. If $\|\mu\| = 0$ then for each $x \in X$, $0 = 2\langle \mu, f_x \rangle - \langle \mu, e \rangle$. Thus, $\langle \mu, e \rangle = 2\mu(x)$ or that μ is constant on X, which implies $\langle \mu, e \rangle = 0 = \mu(x)$. Clearly, for μ_1, μ_2 in $\mathscr{W}$ and scalar α, $\|\mu_1 + \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$ and $\|\alpha\mu_1\| = |\alpha| \|\mu_1\|$. When $\Omega \neq \phi$ and $\omega \in \Omega$, $0 \leq \langle \omega, f_A \rangle \leq 1$ implies $\|\omega\| \leq 1$ and $\langle \omega, e \rangle = 1$ implies $\|\omega\| = 1$. Additivity of the norm follows: $\mu_1, \mu_2 \in \mathscr{C}$ implies (Proposition 2) there exist scalars $\alpha, \beta \geq 0$ and $\omega, \nu \in \Omega$ such that $\mu_1 = \alpha\omega$ and $\mu_2 = \beta\nu$ and thus

$$\|\mu_1 + \mu_2\| = (\alpha + \beta)\left\|\frac{\alpha}{\alpha+\beta}\omega + \frac{\beta}{\alpha+\beta}\nu\right\| = \alpha + \beta = \|\mu_1\| + \|\mu_2\|$$

If $\Omega \neq \phi$, Archimedean order follows directly: $\mu, \nu \in \mathscr{W}$, $\nu \geq 0$, and $\alpha > 0$ in $\mathbb{R}$ with $\alpha\mu \leq \nu$. Then for each $x \in X$, $\langle \alpha\mu, f_x \rangle \leq \langle \nu, f_x \rangle$ and $\langle \nu, f_x \rangle \geq 0$. Thus $\langle \mu, f_x \rangle \leq 0$ or $\mu \leq 0$. $\mathscr{C}$ is $w(\mathscr{W}, \mathscr{L})$ closed: Suppose (μ_α) is a net in $\mathscr{C}$ which converges to $\mu \in \mathscr{W}$ in the $w(\mathscr{W}, \mathscr{L})$ topology. Then for each $A \in \mathscr{E}(\mathscr{A})$,

$0 \le \langle \mu_\alpha, f_A \rangle \to \langle \mu, f_A \rangle$ and $\mu \in \mathscr{C}$. Since the normed topology is stronger than $w(\mathscr{W}, \mathscr{L})$, $\mathscr{C}$ is norm closed.

Completeness: For each $\mu \in \mathscr{W}$, $\|\mu\| \ge |\langle \mu, e \rangle| = |K_\mu|$ and for each $A \in \mathscr{E}(\mathscr{A})$, $2|\langle \mu, f_A \rangle| \le |2\langle \mu, f_A \rangle - \langle \mu, e \rangle| + |\langle \mu, e \rangle| \le \|\mu\| + |K_\mu| \le 2\|\mu\|$; hence, $|\langle \mu, f_A \rangle| \le \|\mu\|$. Clearly, for $\mu, \nu \in \mathscr{W}$ and α in $\mathbb{R}$, $K_{\mu+\nu} = K_\mu + K_\nu$ and $K_{\alpha\mu} = \alpha K_\mu$. Now, let (μ_n) be a norm-Cauchy sequence in $\mathscr{W}$ and for simplicity, let K_n designate K_{μ_n}. Then $\|\mu_n - \mu_m\| \ge |K_n - K_m|$ implies (K_n) is Cauchy in $\mathbf{R}$, and $|\langle \mu_n - \mu_m, f_A \rangle| \le \|\mu_n - \mu_m\|$ for each $A \in \mathscr{E}(\mathscr{A})$ implies $(\langle \mu_n, f_A \rangle)$ is also Cauchy in $\mathbf{R}$. Therefore, define $K = \text{limit}(K_n)$ and define $\mu: \mathscr{E}(\mathscr{A}) \to \mathbf{R}$ by $\langle \mu, f_A \rangle = \text{limit}\langle \mu_n, f_A \rangle$. Since each $x \in X$ implies $\{x\} \in \mathscr{E}(\mathscr{A})$, we can consider $\mu \in \mathbb{R}^X$. We show $\mu \in \mathscr{W}$ and that $(\mu_n) \to \mu$ in norm. To establish (i) of Definition 1 for μ, let E be arbitrary in $\mathscr{A}$ and let A be an arbitrary finite event in E. Since (μ_n) is norm Cauchy, there exists a constant $M > 0$ such that $\|\mu_n\| \le M$ for all n. Since A is finite, there exists an index N such that $n > N$ implies $\Sigma_A |\mu(x) - \mu_n(x)| \le M$. Thus

$$|\Sigma_A \mu(x)| \le \Sigma_A |\mu(x) - \mu_n(x)| + |\langle \mu_n, f_A \rangle| \le 2M$$

Since A was an arbitrary finite subset of E, $\Sigma_E |\mu(x)| \le 4M$.

To obtain (ii), choose E arbitrarily in $\mathscr{A}$ and let $\varepsilon > 0$ be given. There exists an integer N such that $n, m > N$ implies $\|\mu_n - \mu_m\| < \varepsilon/9$ and, in particular, $|K_m - K_n| < \varepsilon/9$; hence, $|K_n - K| \le \varepsilon/9$. Fix $m > N$; then there exists a finite set $A_0 \subset E$ such that for any finite set A with $A_0 \subset A \subset E$, $|\Sigma_A \mu_m(x) - K_m| = |\langle \mu_m, f_{E\setminus A} \rangle| < \varepsilon/9$. Therefore, with $A_0 \subset A$ and any $n > N$, $|\Sigma_A \mu_n(x) - K_n| \le |\Sigma_A \mu_n(x) - \mu_m(x)| + |\Sigma_A \mu_m(x) - K_m| + |K_m - K_n| \le \|\mu_n - \mu_m\| + |\langle \mu_m, f_{E\setminus A} \rangle| + \|\mu_n - \mu_m\| < \varepsilon/3$. Further, since $A_0 \subset A \subset E$ and A is finite, choose $n > N$ and larger if necessary so that $|\Sigma_A \mu(x) - \mu_n(x)| < \varepsilon/3$. Then, $|\Sigma_A \mu(x) - K| \le |\Sigma_A \mu(x) - \mu_n(x)| + |\Sigma_A \mu_n(x) - K_n| + |K_n - K| < \varepsilon$. Thus, $\Sigma_E \mu(x) = K$ and $\mu \in \mathscr{W}$.

Lastly, $(\mu_n) \to \mu$ in norm. For each $\varepsilon > 0$, there exists N such that $n, m > N$ implies $\|\mu_n - \mu_m\| < \varepsilon/4$ and, thus, for each event $A \in \mathscr{E}(\mathscr{A})$, $|\langle \mu_n - \mu_m, f_A \rangle| \le \|\mu_n - \mu_m\| < \varepsilon/4$. Then, for a fixed $m > N$, there exists a finite set $A_0 \subset A$ such that $|\Sigma_A \mu(x) - \mu_m(x)| \le |\Sigma_{A_0} \mu(x) - \mu_m(x)| + \Sigma_{A\setminus A_0} |\mu(x) - \mu_m(x)|$ and $\Sigma_{A\setminus A_0} |\mu(x) - \mu_m(x)| < \varepsilon/4$ (both $\mu, \mu_m \in \mathscr{W}$). Since A_0 finite, we can choose $n > N$ sufficiently large that $|\Sigma_{A_0} \mu(x) - \mu_m(x)| \le |\Sigma_{A_0} \mu(x) - \mu_n(x)| + |\Sigma_{A_0} \mu(x) - \mu_m(x)| \le \Sigma_{A_0} |\mu(x) - \mu_n(x)| + \|\mu_n - \mu_m\| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$. Hence, $|\langle \mu - \mu_m, f_A \rangle| < 3\varepsilon/4$; therefore, $\|\mu - \mu_m\| = \sup\{|\langle \mu - \mu_m, 2f_A - e \rangle|: A \in \mathscr{E}(\mathscr{A})\} < 3\varepsilon$. ■

We now state a theorem of Klee (Peressini, 1967, p. 194) which is of central importance here.

5. Theorem. If $\mathscr{S}$ is an ordered Banach space with a closed cone $\mathscr{C}$ and closed unit ball $\mathscr{U}$, then the Minkowski functional of the set

$(\mathcal{U} \cap \mathcal{C}) - (\mathcal{U} \cap \mathcal{C})$ defines a norm on the subspace $\mathcal{C} - \mathcal{C}$ of $\mathcal{S}$ and this new normed topology is complete and stronger than the original topology of $\mathcal{S}$ restricted to $\mathcal{C} - \mathcal{C}$. When $\mathcal{S} = \mathcal{C} - \mathcal{C}$ (i.e., $\mathcal{C}$ is a *generating* cone) the two norms are topologically equivalent.

Returning to our setting, let $\mathcal{V} = \mathcal{C} - \mathcal{C}$ in $\mathcal{W}$ and let $\mathcal{B} = \text{con}(\Omega \cup -\Omega)$ where $\text{con}(A)$ means convex hull of A. Then $\mathcal{B}$ is radial at 0 in $\mathcal{V}$, circled, and convex; the Minkowski functional of $\mathcal{B}$ defines a seminorm on $\mathcal{V}$ which is, in fact, a norm. This norm is commonly called the *base-norm* for $\mathcal{V}$. If $\mathcal{U}$ is the original closed unit ball of $\mathcal{W}$, then $\mathcal{B} \subset (\mathcal{U} \cap \mathcal{C}) - (\mathcal{U} \cap \mathcal{C})$ since $\mathcal{U} \cap \mathcal{C} = \text{con}(\Omega \cup \{0\})$. Also, $\frac{1}{2}[(\mathcal{U} \cap \mathcal{C}) - (\mathcal{U} \cap \mathcal{C})] = (\frac{1}{2}\mathcal{U} \cap \mathcal{C}) - (\frac{1}{2}\mathcal{U} \cap \mathcal{C}) \subset \mathcal{B}$. Using Klee's theorem, $(\mathcal{U} \cap \mathcal{C}) - (\mathcal{U} \cap \mathcal{C})$ in $\mathcal{V}$ defines a unit ball for a complete norm on $\mathcal{V}$ which is topologically equivalent to the base-norm. Thus, the base-normed topology on $\mathcal{V}$ is complete and stronger than the original normed topology of $\mathcal{W}$ restricted to $\mathcal{V}$.

6. Theorem. When Ω is nonempty the space $\mathcal{V} \subseteq \mathcal{W}$ with the base-norm topology is a Banach space. Each positive linear functional on $\mathcal{V}$ is continuous and since $\mathcal{C}$ is normal in $\mathcal{V}$ and $\mathcal{W}$, each continuous linear functional is the difference of positive and continuous linear functionals. Finally, if the cone $\mathcal{C}$ is generating in $\mathcal{W}$, the two normed topologies are equivalent.

Proof. The first statement is established in the paragraph above. Since the cone $\mathcal{C}$ is generating and closed in $\mathcal{V}$, (Peressini, 1967, Cor. 2.17c, p. 88) or Kelly et al. (1963, p. 102) confirm that each positive linear functional on $\mathcal{V}$ is base-norm continuous. Since $\mathcal{V}^*$ (the continuous dual of $\mathcal{V}$) is an order-unit space (Alfsen, 1971, p. 69), each continuous linear functional is the difference of positive continuous linear functionals. Normality of $\mathcal{C}$ in $\mathcal{W}$ together with Theorem 23.5 of Kelly et al. (1963, p. 227) verifies that each continuous linear functional on $\mathcal{W}$ is the difference of positive and continuous linear functionals. Lastly, when $\mathcal{C}$ is generating, Klee's theorem yields the equivalence of the topologies. ■

We now present two finite examples which demonstrate how different the spaces $\mathcal{V}$ and $\mathcal{W}$ can be. The first example is called the *bow tie* quasimanual (Foulis and Randall, 1972, p. 1674). $\mathcal{A}$ will consist of two operations $\{x, y, z\}$ and $\{z, u, v\}$ having only the point z in common. If we draw a graph of these points and connect with straight lines the elements which are pairwise orthogonal, we obtain Figure 1. This *orthogonality diagram* illustrates clearly why we call this a *bow tie*! We list the operational propositions $\Pi(\mathcal{A})$ of this quasimanual in Figure 2. The lines in the diagram are determined by event inclusion, e.g., $p(x)$ connected to $p(x, z)$ since $\{x\} \subset \{x, z\}$. Note that $\{x, y\}$ *op* $\{u, v\}$, so they determine the same proposition; similarly, e identifies $p(x, y, z)$ and $p(u, v, z)$ at the top. The convex

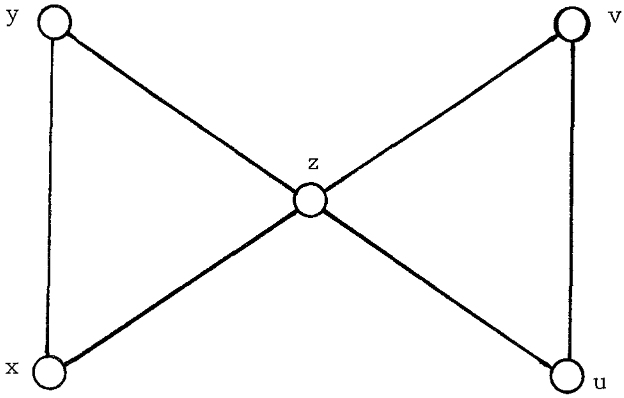

Fig. 1. Bow tie orthogonality diagrams.

set Ω of weights is three-dimensional and its extreme points are the weights $\{\omega_{x,u}, \omega_{x,v}, \omega_{y,u}, \omega_{y,v}, \omega_z\}$ where $\omega_{\alpha,\beta}$ assigns the value 1 to outcomes α, β and 0 to the others and $\omega_z(z)=1$ and 0 otherwise. The space $\mathscr{V}$ is four dimensional with Ω as the base of its cone and $\mathscr{W}=\mathscr{V}$. The set of frequency functionals $\mathscr{L}(\mathscr{A})$ is identified with the extreme points of the order interval $[0, e]$ in $\mathscr{V}^*$ and are in one-to-one correspondence with the elements of $\Pi(\mathscr{A})$. This example suggests that we can identify $\Pi(\mathscr{A})$ with $\mathscr{L}(\mathscr{A})$. This is frequently true, but not always so as the next example indicates.

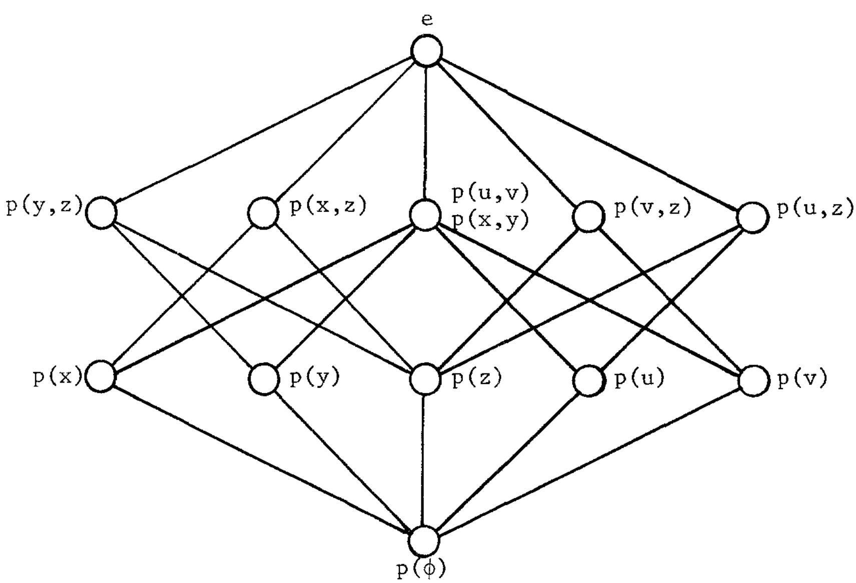

Fig. 2. Bow tie operational logic.

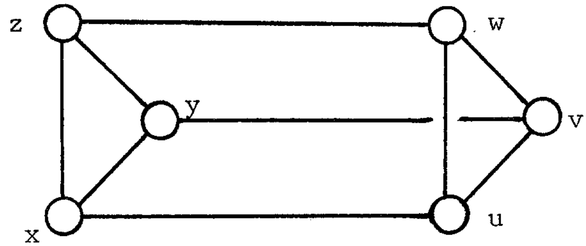

Fig. 3. Orthogonality diagram.

Let $\mathscr{A}$ consist of five operations: $\{x, y, z\}$, $\{u, v, w\}$, $\{x, u\}$, $\{y, v\}$, $\{z, w\}$. We again make the orthogonality diagram as shown in Figure 3. We now determine $\mathscr{W}$ and $\mathscr{V}$. Suppose $\mu \in \mathscr{W}$ and let $\mu(x) = \alpha$ and $\mu(y) = \beta$. Then $\mu(z) = K_\mu - (\alpha + \beta)$, $\mu(u) = K_\mu - \alpha$, $\mu(v) = K_\mu - \beta$ and, therefore, $\mu(w) = \alpha + \beta - K_\mu$; but $K_\mu = \mu(z) + \mu(w) = 0$. If μ were to be in Ω, this would imply $\alpha = \beta = 0$, or that $\Omega = \phi$ and $\mathscr{V} = \{0\}$. Consider μ_1 and μ_2 defined on X:

$$\mu_1(x) = \mu_1(w) = 1/2 \qquad \mu_2(x) = \mu_2(u) = 0$$
$$\mu_1(y) = \mu_1(v) = 0 \qquad \text{and} \qquad \mu_2(y) = \mu_2(w) = 1/2$$
$$\mu_1(z) = \mu_1(u) = -1/2 \qquad \mu_2(z) = \mu_2(v) = -1/2$$

Clearly, μ_1, μ_2 are in $\mathscr{W}$ and if $\mu \in \mathscr{W}$ with $\mu(x) = \alpha$ and $\mu(y) = \beta$ then $\mu = 2(\alpha\mu_1 + \beta\mu_2)$. Further, $\|\mu_i\| = 1$ for $i = 1, 2$. If we let $\mu_3 = \mu_1 - \mu_2$, then we discover that $\pm\mu_1$, $\pm\mu_2$, $\pm\mu_3$ are the extreme points of the hexagonal-shaped unit ball of the two-dimensional space $\mathscr{W}$. Illustrated in Figure 4 are the unit balls of $\mathscr{W}$ and $\mathscr{W}^*$ with labeled extreme points. Note that the frequency functionals of outcomes are in the interior of the unit ball of $\mathscr{W}^*$. Let us now examine the operational logic $\Pi(\mathscr{A})$ for this quasimanual. Each single outcome element is logically equivalent to every other outcome: $\{y\}$ *op* $\{u, w\}$ through $\{v\}$, so $\{u\}, \{w\} \leq \{y\}$; $\{u\}$ *op* $\{z, y\}$ through $\{x\}$, so $\{y\} \leq \{u\}$. Similarly, $\{w\} \leq \{y\}$; hence $\{u\} \equiv \{y\} \equiv \{w\}$. Using the symmetry of the orthogonality diagram we obtain the other equivalences. Likewise, $\{x, y\} \equiv \{x, z\} \equiv \{y, z\} \equiv \{x\} \equiv \{y\} \equiv \{z\} \equiv \cdots$. Tediously continuing we find that $\Pi(\mathscr{A}) = \{p(\phi), p(\{x\}), p(\{x, y, z\})\}$ and thus $\Pi(\mathscr{A})$ is quite different from $\mathscr{L}(\mathscr{A})$. Note that the elements of $\mathscr{W}$ respect the relation *op* but not necessarily $\equiv$. An element of Ω (provided it exists) respects both *op* and $\equiv$.

The following was pointed out to the author by Professor David Foulis.

7. *Proposition.* If $\mathscr{A}$ is a quasimanual with a finite outcome set X and if for each $x \in X$ there exists at least one $\omega_x \in \Omega$ with $\omega_x(x) > 0$, then $\mathscr{W} = \mathscr{V}$.

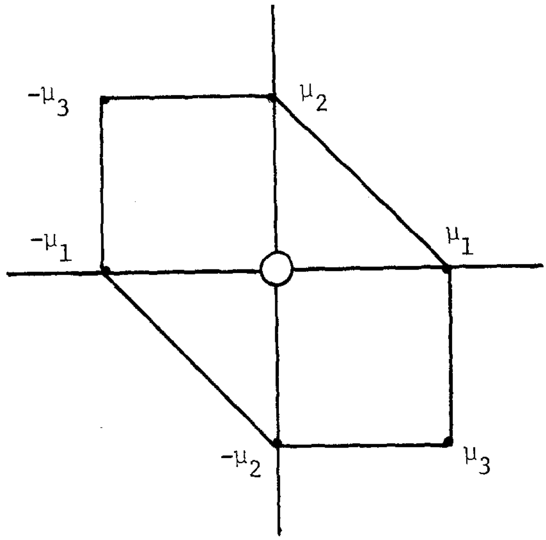

Fig. 4(a). $\mathscr{W}$—Unit ball.

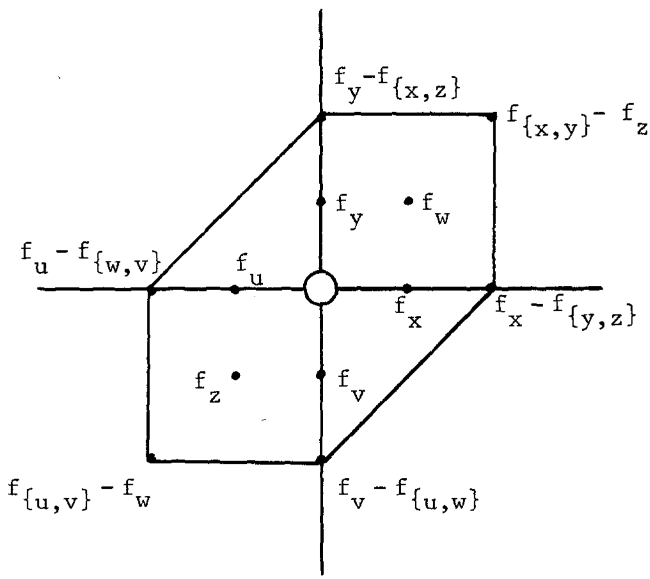

Fig. 4(b). $\mathscr{W}^*$—Unit ball.

Proof. For each $x \in X$, choose a scalar α_x such that $0 < \alpha_x < 1$ and $\Sigma_X \alpha_x = 1$. Then $\omega = \Sigma_X \alpha_x \omega_x \in \Omega$. Let $\mu \in \mathscr{W}$ and since X is finite and ω is positive on X, there exists $\alpha > 0$ such that $\alpha\omega > \mu$ in $\mathscr{W}$. Hence, $\alpha\omega - \mu \in \mathscr{C}$ or there exist $\beta > 0$ and $\nu \in \Omega$ such that $\mu = \alpha\omega - \beta\nu \in \mathscr{V}$. ■

Let us now explain a couple of well-known examples in the formalism of this paper. Let (S, Σ) be a measurable space and we define a quasimanual for (S, Σ) as follows: An operation E will be a countable family (A_n) of pairwise disjoint elements of Σ such that $\bigcup_n A_n = S$. The quasimanual $\mathscr{A}$ is the family of all such operations E and the set of outcomes X for $\mathscr{A}$ is Σ. Notice that classical events of (S, Σ) have been promoted to outcomes for this example. Events for $\mathscr{A}$ are now disjoint collections from Σ. The space $\mathscr{W}$ is readily identified as the collection of all finite signed countably additive measures on (S, Σ). From the well-known Hahn–Jordan decomposition theorem (Dunford and Schwartz, 1958, p. 130) for signed measures, we have that $\mathscr{V} = \mathscr{W}$, i.e., each finite signed measure is the difference of positive measures. The norm of $\mathscr{W}$ and the norm of $\mathscr{V}$ are here in numerical agreement and $\mathscr{V}$ is a Banach lattice in its norm, in fact, it is an L space (Kelly et al., 1963, p. 238).

The second example is the standard model for quantum mechanics. Let H be a separable Hilbert space and we define an operation here as a maximal orthogonal sequence of one-dimensional projections P ($P^2 = P = P^*$) on H. The quasimanual $\mathscr{A}$ is the collection of all such operations, the outcomes are the one-dimensional projections, and the events are all the projections on H. Using Gleason's theorem (1957), we identify Ω as the set of positive trace class operators on H with unit trace and $\mathscr{V}$ as all the self-adjoint trace class operators. The base-norm on $\mathscr{V}$ is the trace norm.

The general question of when $\mathscr{V} = \mathscr{W}$ seems to be a very interesting one. For example, can $\mathscr{V}$ be dense in $\mathscr{W}$ with the relative topology of $\mathscr{W}$ and not equal $\mathscr{V}$? If $\mathscr{V}$ is not dense in $\mathscr{W}$, is the closure of $\mathscr{V}$ complemented in $\mathscr{W}$? If all positive linear functionals are continuous on $\mathscr{W}$, what can be said about $\mathscr{V}$ in $\mathscr{W}$? Many more infinite interesting examples are needed here!

3. NORMAL WEIGHTS AND WEAK COMPACTNESS IN $\mathscr{V}$ AND $\mathscr{W}$

In this section we define and characterize normal functionals and describe the weakly compact subsets of $\mathscr{W}$. Henceforth, we understand that when we refer to $\mathscr{W}$ with its norm, we mean the original norm defined on $\mathscr{W}$ (Definition 3) and, likewise, when we refer to $\mathscr{V}$ with its norm, we mean the base-norm. When $g \in \mathscr{V}^*$ and $\mu \in \mathscr{V}^{**}$ we will write the dual pairing of g and μ as $\langle g, \mu \rangle$. We observe that a subset B of $\mathscr{W}$ is bounded in $\mathscr{W}$ iff there exists a constant $M>0$ such that $\sup_{\mu\in B}\{|\langle\mu, f_A\rangle|: A\in\mathscr{E}(\mathscr{A})\}\leq M$. This follows since the unit ball of $\mathscr{W}$ is the polar of $\{f_A - f'_A: A\in\mathscr{E}(\mathscr{A})\}$.

Recall that a net $(g_\alpha: \gamma\in\Gamma)$ in a partially ordered space is called *monotone increasing* if for $\alpha\leq\beta$ in Γ we have $g_\alpha\leq g_\beta$. The following proposition is essentially Lemma 24.10 of Kelly et al. (1963, p. 243).

8. Proposition. A positive monotone increasing net in $\mathscr{V}$ (respectively $\mathscr{V}^*$) which is bounded above converges in the norm topology (the $w(\mathscr{V}^*, \mathscr{V})$ topology) to its least upper bound.

Proof. Let $(\mu_\gamma: \gamma\in\Gamma)$ be a monotone increasing net in $\mathscr{C}\subset\mathscr{V}$ with $\mu_\gamma\leq\nu\in\mathscr{C}$ for all γ in Γ. Fix γ in Γ; then the net $\{\|\mu_\alpha-\mu_\gamma\|: \alpha\in\Gamma\}$ is monotone increasing and bounded above in $\mathbb{R}$. For $\gamma\leq\alpha\leq\beta$, additivity of the norm gives $\|\mu_\beta-\mu_\alpha\|+\|\mu_\alpha-\mu_\gamma\|=\|\mu_\beta-\mu_\gamma\|\Rightarrow 2\|\nu\|\geq\|\mu_\beta-\mu_\gamma\|\geq\|\mu_\alpha-\mu_\gamma\|$. Hence, $(\mu_\gamma: \gamma\in\Gamma)$ is norm Cauchy and since $\mathscr{V}$ is a Banach space with a closed cone there exists $\mu\in\mathscr{C}$ such that $(\mu_\gamma)\to\mu$ in norm. Further, μ is the supremum of $(\mu_\gamma: \gamma\in\Gamma)$ in $\mathscr{C}$ since $\langle\mu_\gamma, f_x\rangle\to\langle\mu, f_x\rangle$ for each $x\in X$.

Let $(g_\gamma: \gamma\in\Gamma)$ be a monotone increasing net of positive functionals in $\mathscr{V}^*$ which is bounded above by $f\in\mathscr{V}^*$. Define a function g on $\mathscr{C}$ as follows: For $\mu\in\mathscr{C}$, let $0\leq\langle\mu, g\rangle=\lim_\gamma\langle\mu, g_\gamma\rangle\leq\langle\mu, f\rangle$. If $\mu, \nu\in\mathscr{C}$ and $\alpha, \beta\geq 0$ in $\mathbb{R}$, then $\langle\alpha\mu+\beta\nu, g\rangle=\alpha\langle\mu, g\rangle+\beta\langle\nu, g\rangle$ follows directly. If $\mu\in(-\mathscr{C})$, we define $\langle\mu, g\rangle=-\langle-\mu, g\rangle$. We extend g to $\mathscr{V}$ in the obvious fashion: Let $\eta\in\mathscr{V}$; there exists $\mu, \nu\in\mathscr{C}$ such that $\eta=\mu-\nu$ and we define $\langle\eta, g\rangle=\langle\mu, g\rangle-\langle\nu, g\rangle$. Next g is well defined on $\mathscr{V}$, for if in addition $\eta=\bar{\mu}-\bar{\nu}$, $\bar{\mu}, \bar{\nu}\in\mathscr{C}$, then $\mu+\bar{\nu}=\bar{\mu}+\nu$ in $\mathscr{C}$; thus $\langle\mu, g\rangle-\langle\nu, g\rangle=\langle\bar{\mu}, g\rangle-\langle\bar{\nu}, g\rangle$. Clearly, g is linear and for each $\omega\in\Omega$, $0\leq\langle\omega, g\rangle\leq\langle\omega, f\rangle\Rightarrow\|g\|\leq\|f\|$. Since $\langle\omega, g\rangle=\lim_\gamma\langle\omega, g_\gamma\rangle$ for each $\omega\in\Omega$, g is the $\sup\{g_\gamma: \gamma\in\Gamma\}$ in $\mathscr{V}^*$. ■

Following the definition of normal linear functionals for w^* algebras given in Sakai (1971, p. 28), we say that a positive linear functional $\mu\in\mathscr{V}^{**}$ is *normal* provided for each monotone increasing net $(g_\gamma: \gamma\in\Gamma)$ of positive elements in $\mathscr{V}^*$ which is bounded above, $\sup_\Gamma\langle g_\gamma, \mu\rangle=\langle\sup_\Gamma(g_\gamma), \mu\rangle$. An arbitrary element in $\mathscr{V}^{**}$ is called normal provided it can be written as the difference of positive normal functionals.

9. Proposition. When $\mathscr{V}$ is canonically embedded in $\mathscr{V}^{**}$, $\mathscr{V}$ is exactly the set of normal functionals on $\mathscr{V}^*$.

Proof. By the second part of Proposition 8, a monotone increasing net $(g_\gamma: \gamma\in\Gamma)$ of positive elements in $\mathscr{V}^*$ which is bounded above converges w^* to its supremum in $\mathscr{V}^*$; so for each $\mu\in\mathscr{C}$, $\sup_\Gamma\langle\mu, g_\gamma\rangle=\langle\mu, \sup_\Gamma(g_\gamma)\rangle$. Since $\mathscr{C}$ is generating in $\mathscr{V}$, all the elements of $\mathscr{V}$ are normal.

Conversely, suppose μ is positive in $\mathscr{V}^{**}$ and normal. Choose $E\in\mathscr{A}$ and observe that $(f_A: A\subset E,\ A \text{ finite})$ is a monotone increasing net in $\mathscr{V}^*$

of positive elements such that the w^* limit $(f_A: A \subset E, A \text{ finite}) = e \in \mathscr{V}^*$. Thus for each finite event $A \subset E$: $\lim_A \Sigma_A \langle f_x, \mu \rangle = \langle \sup(f_A), \mu \rangle = \langle e, \mu \rangle$. If we define $\mu(x) = \langle f_x, \mu \rangle$ for each $x \in X$, we see that μ is a positive function on X which is also an element of $\mathscr{C}$ in $\mathscr{V}$. Since $\mathscr{V}^{**}$ is also a base-normed space, the elements of $\mathscr{V}^{**}$ which are normal are in $\mathscr{V}$. ■

We now proceed to show that among partially ordered Banach spaces the space $\mathscr{V}$ is the unique predual of $\mathscr{V}^*$.

10. Lemma. Let $\mathscr{M}$ be a partially ordered Banach space with a closed generating cone K. Suppose K^* the dual cone in $\mathscr{M}^*$ contains an order unit e. Then $\mathscr{M}^*$ can be given an order-unit-norm and with this norm $\mathscr{M}^*$ is linearly order homeomorphic to $\mathscr{M}^*$ with its original dual norm. The cone K in $\mathscr{M}$ has a closed base such that the base-norm on $\mathscr{M}$ is equivalent to the original norm of $\mathscr{M}$ and the order-unit dual of $\mathscr{M}$ with this base-norm is $\mathscr{M}^*$ with the order-unit-norm given by the unit e.

Proof. $K^* = \{\phi \in \mathscr{M}^*: \phi(K) \geq 0\}$ is a cone since K is generating in $\mathscr{M}$ and K^* is closed in the dual norm of $\mathscr{M}^*$. Let $I = [-e, e] = \{\phi \in \mathscr{M}^*: -e \leq \phi \leq e\}$. Then $I = (-e + K^*) \cap (e - K^*)$, so I is norm closed, convex, circled, and radial at zero. Since $\mathscr{M}^* = \bigcup_{n=1}^{\infty} nI$, the Baire category theorem yields the existence of $\alpha > 0$ such that $\alpha \mathscr{U}^\circ \subset I$ where $\mathscr{U}$ is the original closed unit ball of $\mathscr{M}$. If $m \in \mathscr{M}$ then $m = m_1 - m_2$, $m_1, m_2 \in K$ and for each $\phi \in I$: $|\langle m, \phi \rangle| \leq \langle m_1, e \rangle + \langle m_2, e \rangle < \infty$; so I is $w(\mathscr{M}^*, \mathscr{M})$ bounded. The uniform boundedness principal yields the existence of $\beta > 0$ such that $\beta \mathscr{U}^\circ \supset I$. Therefore we can introduce the order-unit-norm on $\mathscr{M}^*$ given by e and this order-unit-norm topology is linearly order equivalent to the original dual norm topology on $\mathscr{M}^*$. Let $H = \{m \in \mathscr{M}: \langle m, e \rangle = 1\}$ and let $B = H \cap K$; B is closed and convex and is easily checked to be a base for K. Since $B \subset I_\circ$, I is norm bounded in the original norm of $\mathscr{M}$, further, $\operatorname{con}(B \cup -B)$ is convex, circled, radial at zero, and $w(\mathscr{M}, \mathscr{M}^*)$ bounded. Thus we can introduce the base-norm on $\mathscr{M}$ given by B as a base for K. Let $\mathscr{U}_B = \overline{\operatorname{con}}(B \cup -B)$, where $\overline{\operatorname{con}}$ means the closure in this new normed topology. Clearly, $\mathscr{U}_B \subset I_\circ$. If $m \in I_\circ \backslash \overline{\operatorname{con}}(B \cup -B)$ in $\mathscr{M}$ then there exists $\phi \in \mathscr{M}^*$ such that $\phi(m) > 1$ and $|\phi(\overline{\operatorname{con}}(B \cup -B))| \leq 1$. This is a standard separation theorem; see, for example, Kelly et al. (1963, 14.2, p. 118). So $\phi \notin I$ and therefore its order-unit-norm $\|\phi\|_e > 1$. $\mathscr{M}^{**}$ can be given a base-normed topology with a $w(\mathscr{M}^{**}, \mathscr{M}^*)$-compact base $\tilde{B}$ for which B is w^* dense in $\tilde{B}$ (Alfsen, 1971, p. 78); in fact, $\tilde{B}$ is the $w(\mathscr{M}^{**}, \mathscr{M}^*)$ compactification of B embedded in $\mathscr{M}^{**}$. But $\|\phi\|_e = \sup\{|\langle \phi, \Phi \rangle|: \Phi \in \tilde{B} \text{ in } \mathscr{M}^{**}\} > 1$. Thus there exists $\Phi_0 \in \tilde{B}$ with $|\langle \phi, \Phi_0 \rangle| > 1$ and there also exists a net (m_α) in B such that $(m_\alpha) \to \Phi_0$ in $w(\mathscr{M}^{**}, \mathscr{M}^*)$ topology. However, $|\langle m_\alpha, \phi \rangle| \leq 1$ for all α, so $\langle \phi, \Phi_0 \rangle \leq 1$. Thus no such $m \in I_\circ \backslash \overline{\operatorname{con}} \langle B \cup -B \rangle$ can exist and we have $I_\circ = \overline{\operatorname{con}}(B \cup -B)$ in $\mathscr{M}$. ■

The next theorem shows that $\mathcal{V}$ with the base-norm is linearly order homeomorphic to any partially ordered Banach space with a closed generating cone which acts as a predual of $\mathcal{V}^*$. This is very similar to the result for von Neumann algebras as given in Sakai (1971, p. 30). It would be interesting if the hypothesis of the partial order on the predual could be relaxed, but this author sees no apparent way to do this.

11. Theorem. Let $\mathcal{M}$ be a partially ordered Banach space with a closed generating cone K and suppose $\mathcal{M}^*$ with its dual cone K^* is linearly order homeomorphic with $\mathcal{V}^*$. Then $\mathcal{M}$ is linearly order homeomorphic to $\mathcal{V}$.

Proof. For simplicity, we identify $\mathcal{M}^*$ with $\mathcal{V}^*$ and K^* with the cone of $\mathcal{V}^*$. Then K^* contains the order-unit e of our set $\mathcal{L}$ of frequency functionals. The previous lemma asserts that $\mathcal{M}$ can be equivalently renormed as a base-normed space with the same cone and norm closed unit ball $\mathcal{U}=\overline{\text{con}}(B\cup -B)$. Since $\mathcal{U}^\circ = I$ is $w(\mathcal{V}^*, \mathcal{M})$ compact and I is also $w(\mathcal{V}^*, \mathcal{V})$ compact (Banach–Alaoglu theorem), both w^* topologies must be equivalent on I. If $E\in\mathcal{A}$ then the net $\{f_A: A\subset E,\ A \text{ finite}\}$ converges $w(\mathcal{V}^*, \mathcal{V})$ to $e\in I$ and so must also converge $w(\mathcal{V}^*, \mathcal{M})$ to e. Thus for each $m\in K$, $\lim_A \langle m, f_A\rangle = \langle m, e\rangle$ and m must be in $\mathcal{V}$. Since K generates $\mathcal{M}$, $\mathcal{M}\subset\mathcal{V}$. We may now consider $\mathcal{M}$ as a subspace of $\mathcal{V}^{**}$. Since $w(\mathcal{V}^*, \mathcal{V})$ and $w(\mathcal{V}^*, \mathcal{M})$ agree on I, each member of $\mathcal{V}$ when restricted to I is $w(\mathcal{V}^*, \mathcal{M})$ continuous. So by Grothendieck's completeness theorem (Kelly et al., 1963, p. 145) or (Ringrose, (iv), p. 315), each member of $\mathcal{V}$ must be in the norm completion of $\mathcal{M}$ in $\mathcal{V}^{**}$. Since $\mathcal{M}$ is already a Banach space, $\mathcal{V}\subset\mathcal{M}$. Thus $\mathcal{M}=\mathcal{V}$ and $\mathcal{M}$ in its original topology is linearly order homeomorphic to $\mathcal{V}$ in the base-normed topology. ■

We ask the reader to recall the following lemma concerning weak sequential convergence in the Banach space $l_1(E)$ of all absolutely convergent real-valued sequences on the set E. For a proof see, for instance, Banach (1932, p. 137) or Cook (1978b, p. 277). The technique of proof is sometimes referred to as the *sliding bump.*

12. Lemma. Let $F\subset l_\infty(E)=l_1(E)^*$ be the subspace of functions on E each with finite range. Then a sequence (μ_n) in $l_1(E)$ converges $w(l_1(E), F)$ to $\mu\in l_1(E)$ iff (μ_n) converges to μ in norm.

The following proposition and theorem were established in Cook (1978b) with considerable additional hypotheses. With the results of the previous section, they are obtained far more easily. Independently, Dvurečenskij (1978, p. 292) arrived at roughly the same results; however, the proof given here using Lemma 12 is entirely different. These results are our interpretation of the well-known Hahn–Vitali–Saks theorem of classical measure theory (Dunford and Schwartz, 1958, pp. 158–160) for quasimanuals of operations.

13. Theorem. Suppose (μ_n) is a bounded sequence in $\mathscr{W}$ and suppose $\lim_n \langle \mu_n, f_A \rangle = \mu(A)$ exists for each $A \in \mathscr{E}(\mathscr{A})$. Then $\mu \in \mathscr{W}$.

Proof. Without loss, assume $\|\mu_n\| \leq 1$ for all n. We can consider $\mu: X \to \mathbb{R}$, since $\mu(x) = \lim_n \langle \mu_n, f_x \rangle$ for each $x \in X$. We next observe that μ is absolutely summable on each operation. Choose any $E \in \mathscr{A}$ and any finite event $A \subset E$. Let $\varepsilon > 0$ be given. Then there exists n such that $|\Sigma_A \mu(x)| \leq |\Sigma_A \mu(x) - \mu_n(x)| + |\langle \mu_n, f_A \rangle| < \varepsilon + 1$. Thus $|\Sigma_A \mu(x)| \leq 1$ for all finite events A in E and, therefore, $\Sigma_E |\mu(x)| \leq 2$. Next, μ assigns the same value to each operation: For each $E \in \mathscr{A}$, $\mu(E) = \lim_n \langle \mu_n, f_E \rangle$. Lastly, we must show $\{\Sigma_A \mu(x)\colon A \subset E,\ A \text{ finite}\}$ converges to $\mu(E)$. Fix $E \in \mathscr{A}$ and let each μ_n and μ restricted to E be denoted, respectively, ν_n and ν. Then $\nu_n, \nu \in l_1(E)$ and consider the span $\{f_A\colon A \subset E\} = F_E \subset l_\infty(E)$. Since $\lim_n \langle \nu_n, f_A \rangle = \langle \nu, f_A \rangle$ for each $A \subset E$, the lemma guarantees that $\|\nu_n - \nu\| \to 0$ in $l_1(E)$. Thus for $\varepsilon > 0$, we can choose N such that $n, m > N$, $\Sigma_E |\nu_n(x) - \nu_m(x)| < \varepsilon/3$. Fix $n > N$; there exists a finite event $A_0 \subset E$ such that for any finite event $A \supset A_0$, $|\Sigma_A \nu_n(x) - \nu_n(E)| = |\langle \nu_n, f_{E\setminus A} \rangle| < \varepsilon/3$. For any $m > N$, $|\nu_m(E) - \Sigma_A \nu_m(x)| \leq |\nu_m(E) - \nu_n(E)| + |\nu_n(E) - \Sigma_A \nu_n(x)| + |\Sigma_A \nu_n(x) - \Sigma_A \nu_m(x)| \leq \Sigma_E |\nu_m(x) - \nu_n(x)| + |\langle \nu_n, f_{E\setminus A} \rangle| + \Sigma_E |\nu_n(x) - \nu_m(x)| < \varepsilon$. Holding $A \supset A_0$ fixed, $\lim_m |\nu_m(E) - \Sigma_A \nu_m(x)| = |\nu(E) - \Sigma_A \nu(x)| \leq \varepsilon$. Hence, $\mu \in \mathscr{W}$. ■

14. Theorem. If B is a bounded set in $\mathscr{W}$ then the weak closure of B is $w(\mathscr{W}, \mathscr{W}^*)$ compact iff each sequence (μ_n) in B contains a subsequence (μ_{n_k}) such that $\lim_k \langle \mu_{n_k}, f_A \rangle$ exists for each $A \in \mathscr{E}(\mathscr{A})$.

Proof. $\mathscr{W}$ is a Banach space, so if the closure of B is $w(\mathscr{W}, \mathscr{W}^*)$ compact, by Eberlein's theorem (Schaefer, 1966, p. 185) the closure of B is weakly sequentially compact and the condition follows.

Conversely, it is sufficient to prove each sequence in B has a weakly convergent subsequence. If (μ_n) is in B then by hypothesis we have a subsequence (μ_{n_k}) such that $\lim_k \langle \mu_{n_k}, f_A \rangle = \mu(A)$ exists for each $A \in \mathscr{E}(A)$; thus by Theorem 13, $\mu \in \mathscr{W}$. Since B is weakly bounded in $\mathscr{W}$, it is weakly precompact (Schaefer, 1966, Cor. 2, p. 144); thus, if $\nu \in \mathscr{W}^{**}$ is a $w(\mathscr{W}^{**}, \mathscr{W}^*)$ limit of (μ_{n_k}) in $\mathscr{W}^{**}$, $\langle f_A, \nu \rangle = \langle \mu, f_A \rangle$ for each $A \in \mathscr{E}(\mathscr{A})$. Hence, $\nu = \mu$ in $\mathscr{W}$ and it follows that μ is the $w(\mathscr{W}, \mathscr{W}^*)$ limit of (μ_{n_k}). ■

15. Corollary. The space $\mathscr{W}$ is $w(\mathscr{W}, \mathscr{W}^*)$ sequentially complete.

Proof. If (μ_n) is a weakly Cauchy sequence then it is bounded and for each $A \in \mathscr{E}(\mathscr{A})$, $\lim_n \langle \mu_n, f_A \rangle = \mu(A)$ exists. Theorem 14 guarantees μ is the weak limit of (μ_n) in $\mathscr{W}$. ■

16. Proposition. Each order interval in $\mathscr{W}$ or $\mathscr{V}$ is weakly compact.

Proof. Since the cone $\mathscr{C}$ (in $\mathscr{W}$ or $\mathscr{V}$) is normal (Kelly et al., 1963, 23.7, p. 228), order intervals are bounded and since $\mathscr{C}$ is closed, order intervals are closed. Without loss, we can consider an interval $[0, \nu] \subset \mathscr{C}$. Embed $[0, \nu]$ in $\mathscr{V}^{**}$ and since its $w(\mathscr{V}^{**}, \mathscr{V}^*)$ closure is weakly compact, let $\mu \in \mathscr{V}^{**}$ be a $w(\mathscr{V}^{**}, \mathscr{V}^*)$-accumulation point. We show $\mu \in \mathscr{V}$; to this end there exists a net (μ_α) in $[0, \nu]$ such that $(\mu_\alpha) \to \mu$ in the $w(\mathscr{V}^{**}, \mathscr{V}^*)$ topology. If $E \in \mathscr{A}$ and A is any event in E, then $\lim_\alpha \langle \mu_\alpha, f_A \rangle = \langle f_A, \mu \rangle$. For $\varepsilon > 0$ there exists a finite event $A_0 \subset E$ such that for any finite event A with $A_0 \subset A \subset E$ and all α: $0 \leq \langle \mu_\alpha, f_{E\setminus A} \rangle \leq \Sigma_{E\setminus A} \nu(x) < \varepsilon \Rightarrow 0 \leq \langle f_{E\setminus A}, \mu \rangle \leq \varepsilon$. Since $\langle e, \mu \rangle = \langle f_A, \mu \rangle + \langle f_{E\setminus A}, \mu \rangle$ for each finite event $A \subset E$, the last inequality implies the net $\{\Sigma_A \mu(x);\ A \subset E,\ A \text{ finite}\}$ converges to $\langle e, \mu \rangle$. Thus $\mu \in \mathscr{V}$ and $[0, \nu]$ is $w(\mathscr{V}, \mathscr{V}^*)$ compact in $\mathscr{V}$. ∎

In classical measure theory, Theorem 13 is proved by turning the measurable space into a complete metric and using the Baire category theorem in a very clever way. Since our $\mathscr{E}(\mathscr{A})$ does not form a distributive lattice, this method fails at the outset. Further, the proof given here does not depend on already knowing the result in the classical case. Theorem 14 for $L_1(S, \Sigma, \mu)$ is essentially Theorems 7, 9, and Lemma 8 of Dunford and Schwartz (1958, pp. 291-293). Corollary 15 is Theorem 6 of the same reference on p. 290 and also can be found in Day (1962, Thm. 4, p. 108) for any abstract L space. Our Proposition 16 is stated as Theorem 4 of Day (1962, p. 108) for abstract L spaces. On p. 309 of Dunford and Schwartz (1958), Theorem 8, called Nikodym's boundedness theorem, states that a set M of countably additive measures on a measurable space (S, Σ) for which there exists $N(A) < \infty$ for each A in Σ and all $\mu \in M$ such that $|\mu(A)| \leq N(A)$ implies there exists a number $N < \infty$ such that $|\mu(A)| \leq N$ for all A in Σ and all μ in M. The proof is surprisingly complicated and depends on making equivalence classes of (S, Σ) into a complete metric space and using the Baire category theorem. This method depends on the distributivity of Σ and this is not generally obtained in our $\mathscr{E}(\mathscr{A})$ and so was carefully avoided in Theorem 13. This suggests a similar theorem and proof might exist for boundedness in our space $\mathscr{W}$. This could be a very interesting theorem.

4. THE ULTRAWEAK TOPOLOGY AND COMPACTNESS

Let $\mathscr{L}_f = \{f_A\colon A \in \mathscr{E}(\mathscr{A}),\ A \text{ finite}\} \subset \mathscr{L}$ designate the set of frequency functionals of finite events and let $\mathscr{F}$ designate the linear span of $\mathscr{L}_f$ in $\mathscr{V}^*$. We call the $w(\mathscr{V}, \mathscr{F})$ topology the *ultraweak* topology for $\mathscr{V}$ in analogy with the ultraweak topology for the trace class operators on Hilbert space [see, for example, Ringrose (1972, p. 320)]. Observe that the $w(\mathscr{V}, \mathscr{F})$ topology is the same as the $w(\mathscr{V}, \mathscr{L}_f)$ topology and the ultraweak topology is the weakest linear topology on $\mathscr{V}$ which makes each f_x continuous for each $x \in X$.

Let us now give some physical intuition as to why one should study this linear topology. The finite events are the only events which a real experimenter in a finite amount of time could observe. Further, when trying to calculate the *state* of a physical system one is confronted with finitely many inaccuracies of measurement. Therefore, a reasonable neighborhood of an idealized state $\omega \in \Omega$ should consist of all *physical states* ν such that $|\nu(x_i) - \omega(x_i)| < \varepsilon_i$, where $\varepsilon_i > 0$, $i = 1, 2, \ldots, n$ are errors and $\{x_1, x_2, \ldots, x_n\}$ is a finite event from one of our idealized experiments $E \in \mathscr{A}$. The collection of all such neighborhoods on $\mathscr{V}$ generates the ultraweak topology. These ideas have been adapted from those given by Gunson (1967, p. 269). The reader might ask: Why not use all the events? Would this not give better results? Perhaps! But the mathematics does not work out so well. We return to this point in a moment.

Most physicists would probably agree that the set Ω should contain pure states, i.e., extreme points. Usually one guarantees their existence by using a compactness condition together with the Krein-Milman theorem. Since Ω consists of completely additive weights, most quantum mechanical models do *not* have Ω compact. It is frequently suggested (Gunson, 1967, p. 269, Ludwig, 1983, p. 57) that Ω be completed in a larger space in order to make it compact. Unfortunately for this presentation, if one does this, Ω will pick up some noncompletely additive states and thus will no longer be contained in our basic space $\mathscr{V}$. In the fundamental example of the trace class operators on Hilbert space, none of this completion business is necessary, in fact, it is highly undesirable as the following propositions and discussion will show. We now need the following definition: A collection Δ of weights ($\subset \Omega$) is called *unital* provided for each $A \in \mathscr{E}(\mathscr{A})$, there exists at least one $\omega \in \Delta$ such that $\langle \omega, f_A \rangle = 1$, equivalently, for each $x \in X$, there exists at least one $\omega_x \in \Delta$ such that $\langle \omega_x, f_x \rangle = 1$.

17. Proposition. If there exists a subset $\Delta \subset \Omega$ which is ultraweakly compact and unital, then each operation $E \in \mathscr{A}$ is finite.

Proof. Suppose $B = \{x_1, x_2, \ldots\}$ is a countably infinite event with $B \subset E \in \mathscr{A}$. For each finite event $A \subset B$, there exists $\omega_A \in \Delta$ such that $\omega_A(A) = 0$ and $\omega_A(B \backslash A) = 1$. By hypothesis the net (ω_A: $A \subset E$, A finite) contains an ultraweakly convergent subnet with limit $\omega \in \Delta$. Without loss, we can consider this subnet to be our original net (ω_A: $A \subset E$, A finite). For each $x \in E \backslash B$ and all finite $A \subset B$, $\omega_A(x) = 0$ and thus $\omega(x) = 0$. Further, for each finite $A_0 \subset E$ and finite A with $A_0 \subset A \subset E$, $\omega_A(A_0) = 0$. Consequently, $\omega(A_0) = 0$ and since ω is completely additive on E, $\omega(E) = 0$; but $\omega \in \Delta$, so $\omega(E) = 1$. Thus no such B exists and each operation in $\mathscr{A}$ is finite. ■

If Ω is $w(\mathscr{V}, \mathscr{F})$ compact then $\mathscr{U} = \text{con}(\Omega \cup -\Omega)$ is also compact. However, $\mathscr{U}$ can be $w(\mathscr{V}, \mathscr{F})$ compact and Ω not: e.g., it is easy to see that when H is separable Hilbert space, Ω being the positive, unit trace operators on H is not $w(\mathcal{V}, \mathcal{F})$ compact, but the unit ball of the trace class is ultraweakly compact. Recall the trace class is the dual of the space of compact operators, which in turn, is the operator norm completion of the finite-dimensional operators. Let us demonstrate that Ω is not ultraweakly compact. If (e_k) $k=1,2,\ldots$ is an orthonormal sequence in H, then each one-dimensional projection $\omega_k: H \to H$ given by $\omega_k(x)=(x, e_k)e_k$ where $(\cdot, \cdot)$ is the inner product of H, is in Ω. Now $(\omega_k) \to 0$ ultra weakly from the lemma on Fourier coefficients in H.

Notice that Ω will have extreme points if the unit ball $\mathcal{U} = \text{con}(\Omega \cup -\Omega)$ of $\mathcal{V}$ is compact for some linear topology since the extreme points of $\mathcal{U}$ lie in $\pm\Omega$. From the quasimanual point of view the weakest linear topology which could make $\mathcal{U}$ compact would be that given by the frequency functionals of outcomes of the underlying quasimanual $\mathcal{A}$. If one used more frequency functionals (a stronger topology) one would still obtain ultraweak compactness. In this light we interpret the following well-known theorem concerning Banach dual spaces; see, e.g., Ringrose (1972, Sections 1, 2). $\mathcal{U}$ is $w(\mathcal{V}, \mathcal{F})$ compact iff $\mathcal{V}$ is norm and order isometric with the space which is dual to the norm completion of $\mathcal{F}$ in $\mathcal{V}^*$. When $\mathcal{U}$ is ultraweakly compact we have a very close analogy with the type-I von Neumann algebra. The following proposition emphasizes the primality of the finite events in a quasimanual when $\mathcal{U}$ is ultraweakly compact.

18. Proposition. If $\mathcal{A}$ is a quasimanual, Δ is a unital set of states in Ω, and the unit ball $\mathcal{U}$ of $\mathcal{V}$ is $w(\mathcal{V}, \mathcal{F})$ compact, then a finite event is not operationally perspective to an infinite event.

Proof. Suppose on the contrary that $A \; op \; B$, A is a finite event, and B is an infinite event. Then there exists an event C which is an operational complement to both A and B. If D represents any finite event in B, then there exists $\omega_D \in \Delta$ such that $\langle \omega_D, f_{B\setminus D}\rangle = 1$. Thus, $\langle \omega_D, f_D\rangle = 0$. If $\mathcal{D}$ represents the family of all finite subsets of B directed by inclusion, then the net $(\omega_D\colon D \in \mathcal{D})$ contains an ultraweakly convergent subnet which has limit $\mu \in \mathcal{U} \cap \mathcal{C}$. Note that $\mathcal{U} \cap \mathcal{C}$ is also compact since it is easy to see that $\mathcal{C}$ is $w(\mathcal{V}, \mathcal{F})$ closed. Without loss we can assume the convergent subnet is $(\omega_D\colon D \in \mathcal{D})$. Now $f_B \geq f_{B\setminus D}$ in $\mathcal{V}^*$ and $\langle \omega_D, f_{B\setminus D}\rangle = 1 \Rightarrow \langle \omega_D, f_B\rangle = 1$ for all $D \in \mathcal{D}$. Since $A \; op \; B$ and each $\omega_D \in \Omega \Rightarrow \langle \omega_D, f_A\rangle = 1$. Consequently, with $f_A \in \mathcal{L}_f \Rightarrow \langle \mu, f_A\rangle = 1$ and thus $\langle \mu, f_B\rangle = 1$. Now fix $b \in B$ and observe for any D in $\mathcal{D}$ with $\{b\} \subset D$ that $\langle \omega_D, f_b\rangle = 0$. Since $f_b \in \mathcal{L}_f$, $\langle \mu, f_b\rangle = 0$. But, $\mu \in \mathcal{U} \cap \mathcal{C}$ so $\langle \mu, f_B\rangle = \mu(B) = \Sigma_B \mu(b) = 0$. ■

The following example illustrates the limited conclusions one can draw from these previous propositions. Let $X = \{a_0, a_1, a_2, \ldots\}$ and let $\mathcal{A} = \{\{a_0, a_1\}, \{a_0, a_2, a_3\}, \{a_0, a_4, a_5, a_6\}, \ldots\}$. Since X is infinite, we construct

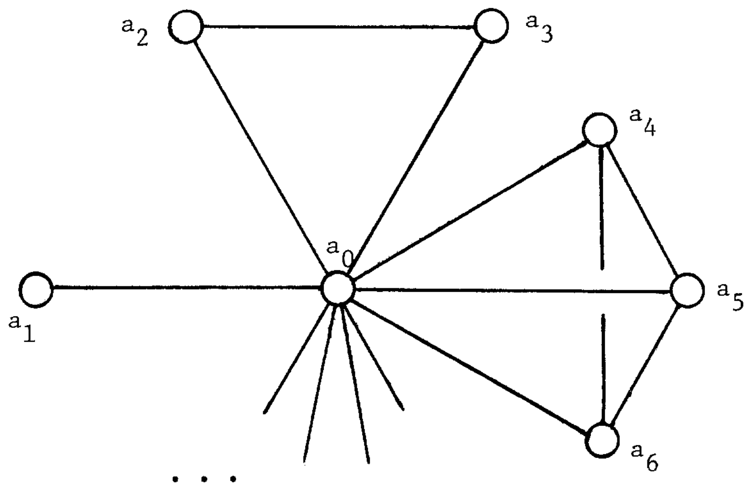

Fig. 5. Partial orthogonality diagram.

a partial orthogonality diagram to illustrate this quasimanual (Figure 5). Then $\{a_1\}$ *op* $\{a_0\}$, $\{a_1\}$ *op* $\{a_2, a_3\}$, $\{a_1\}$ *op* $\{a_4, a_5, a_6\}, \ldots$. One can easily check that Ω is unital and ultraweakly compact for this quasimanual. Observe that each operation in $\mathscr{A}$ is finite, but there is no finite bound on the cardinality of the events *op* to $\{a_1\}$. Further, $p(\{a_1\})$ is not an *atom* in $\Pi(\mathscr{A})$. Since $p(\phi) < p(\{a_2\})$, $p(\{a_4\}), \ldots < p(\{a_1\})$ in $\Pi(\mathscr{A})$—see Randall and Foulis (1983) for further details concerning the order properties of $\Pi(\mathscr{A})$ for any quasimanual $\mathscr{A}$.

In Cook (1978a, Cor. 6) and Gunson (1967, p. 270) weak compactness of the base of the cone was used to prove the space of states is a Banach space. If $\mathscr{V}$ is to have a compact, unital base and for each $x \in X$, f_x is to be continuous for this topology, then Proposition 17 states that one must only have operations which are finite. Independently, G. T. Rüttimann (1981, Theorem 4.2) has come to a similar conclusion. Thus, if one is going to use $\mathscr{V}$ as the predual of $\mathscr{V}^*$ (the space generated by the frequency functionals), then the insistence on a compact base seems not to be a good one. If a major reason to have a compact base is to prove $\mathscr{V}$ is norm complete, this is not necessary either, since norm completeness of $\mathscr{V}$ was obtained in Theorem 6 without this assumption. One last closing remark is in order. If $\mathscr{A}$ has at least one infinite operation then $\mathscr{V}^{**}$ always contains finitely additive states which are not completely additive. If this were not so, Ω would be $w(\mathscr{V}, \mathscr{V}^*)$ compact and thus $w(\mathscr{V}, \mathscr{F})$ compact.

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